

Axiomatization and complexity of modal logic with knowing-what operator on model class \mathbf{K}^*

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Abstract

Standard epistemic logic studies propositional knowledge, yet many other types of knowledge such as “knowing whether”, “knowing what”, “knowing how” are frequently and widely used in everyday life as well as academic fields. In [11], an axiomatization of the epistemic logic with both regular “knowing that” operator and “conditionally knowing what” operator is given. Then the decidability and complexity of this logic command our study. In this paper, we direct our attention to the arbitrary models instead of epistemic models, thus making things clearer, and then give a axiomatization and a tableau for the modal logic with same operators on arbitrary Kripke models. Given the tableau of this logic, the complexity of the satisfiability problem of this logic is **PSPACE**-complete.

Keywords: Knowing what, modal logic, tableau, **PSPACE**-complete

1 Introduction

Standard epistemic logic studies “knowing that” operators, that is, $K_i\phi$ means agent i knows that ϕ is true. While this perspective fixed our focus on *propositional knowledge*, its succinctness facilitated our study and application of it. Recent decades witnessed the prosperity of numerous logics with standard knowing-that operator or similar propositional operators in fields like philosophy, computer science, game theory etc., with Kripke semantics or neighbourhood semantics (see [5, 2, 9] for exemplary works). However, there are also other interesting knowledge expressions used in our everyday life like “knowing whether”, “knowing what” and “knowing how” which have raised many interesting questions in linguistics and philosophy, but received less studies in logic.

Among these ways of expressing knowledge, “knowing what” is particularly suitable for the beginning of our logical study aiming for the vast ocean of non-standard knowledge operators, for it is a richer topic compared to “knowing whether”, less arguable than “knowing how” philosophically, interesting in its own logical and mathematical properties and readily applicable in other fields like cryptography. For example, sentences like “he knows that she knows her private key, but he do not know what exactly his private key is.” are typical in security settings. With propositional knowledge operator K alone, we may have a formula $K_iK_jp \wedge \neg K_ip$ to express this. But by axiom T in standard epistemic logic, this formula is not consistent. Introducing something new is obviously needed and several attempts was made recently, such as [3, 8] in security settings.

In fact, in his grounding work of epistemic logic [4], Hintikka has already briefly discussed “knowing who” in ch.6.3, an operator with evident similarity with “knowing what”, in terms of first-order modal logic. In [7], a seminal work that is hitherto mostly referred to by the studies of Public Announcement Logic, Plaza actually also proposed a “knowing what” operator Kv , of course in the context of Public Announcement Logic. This leaves us a logic with both “knowing what” and public announcement.

Technically, Kv operator packs a existential quantifier with a modality together, and the resulting logic is a (small) fragment of first-order modal logic, which requires new techniques to handle. To deal with public announcement part, we need to change our perspective and pack announcement into “knowing what” operator to make it a conditional one. Thus until in [11, 10] by Wang and Fan did we see a complete axiomatization for the logic of “knowing what” with the power of model relativization, i.e. **ELKv**^r. Because of the application potential of this logic, such as in the field of computer science and AI as argued by McCarthy in [6], the decidability and complexity of this logic become important. In [12], Xiong has shown that **ELKv**^r is decidable for its small model property. As for complexity, this paper serves as a preliminary step.

In this paper, we show that the axiomatization of Wang and Fan without the characteristic S5 axioms is also complete w.r.t. the logic on the class of arbitrary models (call it **LKv**^r, that is, **ELKv**^r without the initial “Epistemic”). We simplified the proof of completeness in [11] significantly. With the constraint of reflexivity, there are some interactions between agents, thus the beautiful property of the conditional part of knowing what operator in one agent is obscured and complicated. Without such constraint, we can work on the knowledge of one agent more easily and abstractly.

Moreover, we show that the complexity of the satisfiability of the logic is **PSPACE**-complete, which is no more complex than normal modal logic. This is by way of a tableau. Normally a tableau means two things: first, to test the satisfiability of a formula, only its subformula counts and thus we can do trials on each of those subformulas by setting it true or false; second, we have a canonical or minimal way to deal with modal operators, much like the spirit of Sahlqvist's minimal assignment method, such that if this minimal way fails, all possible ways fail necessarily. As for our logic on model class **K**, the first property is also true, and for the second property, there is not "a" canonical way but an array of them, enumerable within **PSPACE**.

The rest of this paper is structured as such: we first give the syntax and semantics of **LKv^r** and its proof system **LKV^r** in section 2. Section 3 presents the completeness results and Section 4 the complexity. We then conclude this paper with future work in Section 5.

2 Preliminaries

We follow the notation proposed in [11]. However, since we are now working on arbitrary Kripke models, it is no longer appropriate to use K as modal operator. So we now return to the box and diamond notation.

Given a countably infinite set of proposition letters **P**, a countably infinite set of agent names **I**, and a countably infinite set of (non-rigid) constant symbols **D**, the language of **LKv^r** is defined as follows:

$$\phi ::= \top \mid p \mid \neg\phi \mid (\phi \wedge \phi) \mid \Box_i\phi \mid \nabla_i(\phi, d)$$

where $p \in \mathbf{P}, i \in \mathbf{I}, d \in \mathbf{D}$. Our new operator $\nabla_i(\phi, d)$ here says that, in all possible cases where ϕ is true, the value of d is all the same. For example, the sentence "I know your password if it is a four-digit number" can be expressed as $\nabla(\text{four-digit_number_password}, \text{password})$. As usual, we define $\perp, (\phi \vee \psi), (\phi \rightarrow \psi), \Diamond_i\phi$ as the abbreviations of, respectively, $\neg\top, \neg(\neg\phi \wedge \neg\psi), \neg(\phi \wedge \neg\psi), \neg\Box_i\neg\phi$. Parentheses will be omitted unless confusion arises.

For future convenience, write $Sub(\phi)$ for the set of subformulas of ϕ , where for $\nabla_i(\phi, d)$, its subformulas are itself plus all the subformulas of ϕ . Then define $Sub^+(\phi) = \{\neg\phi \mid \phi \in Sub(\phi)\} \cup Sub(\phi)$. Let $D(\phi)$ be the set of value name that occurs in ϕ . At the same time we need $depth(\phi)$ denoting the modal depth of ϕ . For new operator ∇_i , we define $depth(\nabla_i(\phi, d)) = depth(\phi) + 1$. Further, for any finite set of formula X :

$$\begin{array}{ll} Sub(X) &= \bigcup_{\phi \in X} Sub(\phi) & \neg X &= \{\neg\phi \mid \phi \in X\} \\ Sub^+(X) &= \bigcup_{\phi \in X} Sub^+(\phi) & X \setminus \Box_i &= \{\phi \mid \Box_i\phi \in X\} \\ D(X) &= \bigcup_{\phi \in X} D(\phi) & \phi_X &= \bigwedge_{\phi \in X} \phi \\ depth(X) &= \max\{depth(\phi) \mid \phi \in X\} & \Box_i X &= \{\Box_i\phi \mid \phi \in X\} \end{array}$$

To interpret **LKv^r**, we need to extend common Kripke models to incorporate the assignment of the names of value in **D**, and this can also be seen as a first-order Kripke model with a constant domain. So a model of **LKv^r** $\mathcal{M} = \langle S, O, \{\rightarrow_i \mid i \in \mathbf{I}\}, V, V_D \rangle$, in which S is a non-empty set of possible worlds, O is a non-empty set of values, \rightarrow_i is a binary relation on S , V is a function assigning each proposition letter $p \in \mathbf{P}$ a set of possible world $V(p) \subset S$ where p is true, and V_D

a function from $\mathbf{D} \times S$ to O so that each value name $d \in \mathbf{D}$ at each possible world s is assigned a value $V_D(d, s)$. Let \mathbf{K} denote the class of all models defined above. Now the semantics:

$\mathcal{M}, s \models \top$	always holds
$\mathcal{M}, s \models p$	$\Leftrightarrow s \in V(p)$
$\mathcal{M}, s \models \neg \phi$	$\Leftrightarrow \mathcal{M}, s \not\models \phi$
$\mathcal{M}, s \models \phi \wedge \psi$	$\Leftrightarrow \mathcal{M}, s \models \phi$ and $\mathcal{M}, s \models \psi$
$\mathcal{M}, s \models \Box_i \phi$	\Leftrightarrow for all t such that $s \rightarrow_i t : \mathcal{M}, t \models \phi$
$\mathcal{M}, s \models \nabla_i(\phi, d)$	\Leftrightarrow for any $t_1, t_2 \in S$ such that $s \rightarrow_i t_1$ and $s \rightarrow_i t_2$: if $\mathcal{M}, t_1 \models \phi$ and $\mathcal{M}, t_2 \models \phi$, then $V_D(d, t_1) = V_D(d, t_2)$

Intuitively, $\nabla_i(\phi, d)$ is true at s if and only if in all i -accessible worlds where ϕ is true, d is assigned a uniform value. Conversely, if $\nabla_i(\phi, d)$ is false, then there must be two i -accessible ϕ -worlds that disagree on the value of d . From the perspective of first-order modal logic, $\nabla_i(\phi, d)$ can be seen as $\exists x \Box_i(\phi \rightarrow d = x)$ where x is a rigid variable and c is a non-rigid one. Thus a ∇ is actually a package consists of a quantifier, a modality, a implication and a equality.

As for the derivation system, it is enough to just exclude axioms particular to S5 from the system proposed in [11]:

System \mathbf{LKV}^r

Axiom Schemas

TAUT	all the instances of tautologies
K	$\Box_i(\phi \rightarrow \psi) \rightarrow (\Box_i \phi \rightarrow \Box_i \psi)$
DISTV	$\Box_i(\phi \rightarrow \psi) \rightarrow (\nabla_i(\psi, d) \rightarrow \nabla_i(\phi, d))$
$V\perp$	$\nabla_i(\perp, d)$
$V\vee$	$\Diamond_i(\phi \wedge \psi) \wedge \nabla_i(\phi, d) \wedge \nabla_i(\psi, d) \rightarrow \nabla_i(\phi \vee \psi, d)$

Rules

MP	$\frac{\phi, \phi \rightarrow \psi}{\psi}$
NEC	$\frac{\phi}{\Box_i \phi}$
RE	$\frac{\psi \leftrightarrow \chi}{\phi \leftrightarrow \phi[\psi/\chi]}$

3 Completeness

Our proof of the completeness of \mathbf{LKV}^r proceeds in a standard Henkin way: use maximal consistent sets as the basis of the canonical model, link the canonical relations properly so that a existence lemma can be proven, use existence lemma to prove truth lemma and then completeness follows immediately. However, as our ∇ operator packs many things in it, simply a maximal consistent set does not give us enough information to pin down every possibility. Thus we need to saturate these maximal consistent sets consistently. Specifically, since $\nabla_i(\phi, d)$ is actually $\exists x \Box_i(\phi \rightarrow d = x)$, its subformulas $\Box_i(\phi \rightarrow d = x)$ and $d = x$ need their counterpart in canonical model. Now we give the definition:

Definition 1. Denote the set of all maximal consistent sets w.r.t. \mathbf{LKV}^r as \mathbf{MCS} and the set of natural number as \mathbb{N} . The canonical model $\mathcal{M}^c = \langle S^c, \mathbb{N}, \{\rightarrow_i \mid i \in I\}, V^c, V_D^c \rangle$ where¹:

- S^c consists of all the triples $\langle \Gamma, f, g \rangle \in \mathbf{MCS} \times \mathbb{N}^D \times (\mathbb{N} \cup \{*\})^{I \times \mathbf{LKV}^r \times D}$ that satisfy the following two conditions for any $i \in I, \phi, \psi \in \mathbf{LKV}^r, d \in D$:

$$(1) \quad g(i, \phi, d) \neq * \text{ iff } \nabla_i(\phi, d) \wedge \Diamond_i \phi \in \Gamma;$$

¹A countable set \mathbb{N} is already big enough as the constant domain of objects. See footnote in [11]. Also note that following clause (2) is slightly different from clause (ii) in [11].

(2) $g(i, \phi, d) \neq *$ and $g(i, \psi, d) \neq *$ imply: $g(i, \phi, d) = g(i, \psi, d)$ iff $\nabla_i(\phi \vee \psi, d) \in \Gamma$

for any $s \in S^c$, we use Γ_s, f_s, g_s to denote the three components of s and we simplify $\phi \in \Gamma_s$ as $\phi \in s$.

- For $s, t \in S^c$, $s \rightarrow_i t$ iff the following two conditions are satisfied:

(3) $\{\phi \mid \Box_i \phi \in s\} \subseteq t$.

(4) $\nabla_i(\phi, d) \in s$ and $\phi \in t$ imply $f_t(d) = g_s(i, \phi, d)$.

Here, g is the counterpart of $\Box(\phi \rightarrow d = x)$ and f the counterpart of $d = x$. To be explicit about their meaning, $g(i, \phi, d)$ gives the value of d in all the ϕ -worlds accessible by i , and $f(d)$ gives the value of d directly. The star symbol obviously means that if there are no ϕ -worlds accessible from i , $g(i, \phi, d)$ should reflect this fact by a value not in \mathfrak{N} .

Given this canonical model, existence lemma is then our aim. In ordinary model logic, it is enough to use Lindenbaum lemma to extend $\Gamma \setminus \Box_i$ to build a i -successor of Γ . However, as our canonical model requires more information, or a saturation, we must show that such a saturation is possible i.e. is consistent with what we already have. The following proposition states this technically:

Proposition 2. *Given a possible world $s \in S^c$, an agent $i \in I$, a maximal consistent set Γ such that $\{\phi \mid \Box_i \phi \in s\} \subseteq \Gamma$ and a natural number $x \in \mathbb{N}$, we can construct $t = \langle \Gamma, f, g \rangle$ using x such that $t \in S^c$ and $s \rightarrow_i t$.*

Proof. Note that the only thing we need to do is to construct appropriate f and g so that $t = \langle \Gamma, f, g \rangle$ satisfies the requirements (1), (2) and (4) stated in definition 1, since (3) is already satisfied. We first construct f (which is easier) and then g .

For any $d \in \mathbf{D}$:

$$f(d) = \begin{cases} g_s(i, \phi, d) & \text{there exists a } \phi \in \mathbf{LKv}^r : \phi \in \Gamma \text{ and } \nabla_i(\phi, d) \in s \\ x & \text{otherwise} \end{cases}$$

Obviously, if this f is well-defined, then (4) in definition 1 will be satisfied. Now we claim that this definition is indeed well-defined, that is, for any $\phi, \psi \in \mathbf{LKv}^r$ such that $\phi \in \Gamma$, $\nabla_i(\phi, d) \in s$, $\psi \in \Gamma$ and $\nabla_i(\psi, d) \in s$, we have $g_s(i, \phi, d) = g_s(i, \psi, d)$.

First, if $\phi \in \Gamma$ and $\psi \in \Gamma$, then $\Diamond_i(\phi \wedge \psi) \in s$. Suppose not, since s is maximal, $\Box_i(\neg\phi \vee \neg\psi) \in s$. Then $\neg\phi \vee \neg\psi \in \Gamma$. Again, since Γ is maximal, either $\neg\phi \in \Gamma$ or $\neg\psi \in \Gamma$. But either way, Γ will be inconsistent.

Now $\Diamond_i(\phi \wedge \psi), \nabla_i(\phi, d), \nabla_i(\psi, d)$ are all in s . By axiom \mathbf{VV} and the maximality of s , $\nabla_i(\phi \vee \psi, d) \in s$. According to clause (2) of definition 1, $g_s(i, \phi, d) = g_s(i, \psi, d)$ and this concludes the proof of the well-definedness of f .

The construction of g is more involved because of the clause (2). For any $i \in I$ and any $d \in \mathbf{D}$, first we construct a partition on the set $G(i, d) = \{\phi \in \mathbf{LKv}^r \mid \nabla_i(\phi, d) \wedge \Diamond_i \phi \in \Gamma\}$. Note that this set is exactly the collection of formulas that we need to give a non-star value through $g(i, \phi, d)$, and the clause (2) is effective only on this set. For any two $\phi, \psi \in G(i, d)$, let $\phi \sim_{i,d} \psi$ iff $\nabla_i(\phi \vee \psi, d) \in \Gamma$. Now we claim that $\sim_{i,d}$ is an equivalence relation:

Reflexivity For $\phi \in G(i, d)$, by definition of $G(i, d)$, $\nabla_i(\phi, d) \in \Gamma$. By TAUT, $\phi \leftrightarrow \phi \vee \phi$. By RE and the maximality of Γ , $\nabla_i(\phi \vee \phi, d) \in \Gamma$. So $\phi \sim_{i,d} \phi$.

Symmetry For $\phi, \psi \in G(i, d)$, if $\phi \sim_{i,d} \psi$, then $\nabla_i(\phi \vee \psi, d) \in \Gamma$. By TAUT and RE, $\vdash \nabla_i(\phi \vee \psi, d) \leftrightarrow \nabla_i(\psi \vee \phi, d)$. By maximality of Γ , $\nabla_i(\psi \vee \phi, d) \in \Gamma$. So $\psi \sim_{i,d} \phi$.

Transitivity This is more interesting. For $\phi, \psi, \chi \in G(i, d)$, suppose $\phi \sim_{i,d} \psi$ and $\psi \sim_{i,d} \chi$. By definition, we have $\Diamond_i \psi$, $\nabla_i(\phi \vee \psi, d)$ and $\nabla_i(\psi \vee \chi, d)$ all in Γ . Notice that we have following derivation:

[1]	$\psi \rightarrow (\phi \vee \psi) \wedge (\psi \vee \chi)$	TAUT
[2]	$\Diamond_i \psi \rightarrow \Diamond_i((\phi \vee \psi) \wedge (\psi \vee \chi))$	K rule
[3]	$\Diamond_i((\phi \vee \psi) \wedge (\psi \vee \chi)) \wedge \nabla_i(\phi \vee \psi, d) \wedge \nabla_i(\psi \vee \chi, d) \rightarrow \nabla_i((\phi \vee \psi) \vee (\psi \vee \chi), d)$	$\vee\vee$
[4]	$(\phi \vee \chi) \rightarrow (\phi \vee \psi) \vee (\psi \vee \chi)$	TAUT
[5]	$\Box_i((\phi \vee \chi) \rightarrow (\phi \vee \psi) \vee (\psi \vee \chi))$	NEC
[6]	$\nabla_i((\phi \vee \psi) \vee (\psi \vee \chi), d) \rightarrow \nabla_i(\phi \vee \chi, d)$	DISTV, [5]
[6]	$\Diamond_i \psi \wedge \nabla_i(\phi \vee \psi, d) \wedge \nabla_i(\psi \vee \chi, d) \rightarrow \nabla_i(\phi \vee \chi, d)$	[2] [3] [6]

Since Γ is a maximal consistent set w.r.t. \mathbf{LKv}^r , this tells us that $\nabla_i(\phi \vee \chi, d) \in \Gamma$. So $\phi \sim_{i,d} \chi$.

Write $[\phi]_{i,d} = \{\psi \in G(i, d) \mid \psi \sim_{i,d} \phi\}$. Since $\mathbf{I}, \mathbf{P}, \mathbf{D}$ are all countable, \mathbf{LKv}^r is countable, then $G(i, d)$ is countable, so $\{[\phi]_{i,d} \mid \phi \in G(i, d)\}$ is countable. Then there is an injection $h_{i,d}$ from $\{[\phi]_{i,d} \mid \phi \in G(i, d)\}$ to \mathbb{N} . Now we can define g :

$$g(i, \phi, d) = \begin{cases} h_{i,d}([\phi]_{i,d}) & \phi \in G(i, d) \\ * & \text{otherwise} \end{cases}$$

It's now easy to see that this definition satisfies (1) and (2) of definition 1. \square

For future convenience, we call this construction as F , that is, $F(s, \Gamma, a, x) = \langle \Gamma, f, g \rangle$ where f and g are defined as above.

After the above proposition, we are now able to prove existence lemma. First is the existence lemma for $\neg\Box_i\phi$:

Lemma 3. For any $s \in S^c$, any $i \in \mathbf{I}$, any $\phi \in \mathbf{LKv}^r$: $\neg\Box_i\phi \in s$ implies that there is a world $t \in S^c$ such that $s \rightarrow_i t$ and $\neg\psi \in t$.

Proof. It is a standard modal logic exercise to show that $X = \{\neg\psi\} \cup \{\phi \mid \Box_i\phi \in s\}$ is consistent. By Lindenbaum Lemma (for \mathbf{LKv}^r), X can be extended into a **MCS** Γ . Then by proposition 2, Γ can again be extended into a possible world $t = F(s, \Gamma, i, 0) \in S^c$ such that $s \rightarrow_i t$. Since $\neg\psi \in X$, $\neg\psi \in t$. \square

Now we need to deal with formulas in the form of $\neg\nabla_i(\psi, d)$. Following the convention of dealing with $\neg\Box_i\psi$, what we need to do is to show that if $\neg\nabla_i(\psi, d)$ is present in some possible world s , then there are indeed $t_1, t_2 \in S^c$ such that $s \rightarrow_i t_1, s \rightarrow_i t_2$ and $f_{t_1}(d) \neq f_{t_2}(d)$. More specifically:

Lemma 4. For any $s \in S^c$ such that $\neg\nabla_i(\psi, d) \in s$, there exists $t_1, t_2 \in S^c$ such that $\psi \in t_1$, $\psi \in t_2$, $s \rightarrow_i t_1$, $s \rightarrow_i t_2$ and $f_{t_1}(d) \neq f_{t_2}(d)$.

Proof. Suppose $s \in S^c$ and $\neg \nabla_i(\psi, d) \in s$. Now we intend to prove

(!) there exists $t_1, t_2 \in S^c$ such that $\psi \in t_1$, $\psi \in t_2$, $s \rightarrow_i t_1$, $s \rightarrow_i t_2$ and $f_{t_1}(d) \neq f_{t_2}(d)$.

Again we use the notation $G(i, d) = \{\phi \in \mathbf{LKv}^r \mid \nabla_i(\phi, d) \wedge \Diamond_i \phi \in s\}$, $\sim_{i,d} = \{(\phi_1, \phi_2) \in G(i, d)^2 \mid \nabla_i(\phi_1 \vee \phi_2, d) \in s\}$ and $[\phi]_{i,d} = \{\phi' \in G(i, d) \mid \phi \sim_{i,d} \phi'\}$ as defined in proposition 2. Let $A = \{\phi \mid \Box_i \phi \in s\}$, $A^+ = A \cup \{\psi\}$, $\overline{G(i, d)} = \{\neg \chi \mid \chi \in G(i, d)\}$. Note that A^+ is consistent. Suppose it is not, then there is a finite subset B of A such that $\vdash \bigwedge B \rightarrow \neg \psi$. By NEC and distribution of \Box_i , $\vdash \bigwedge \Box_i B \rightarrow \Box_i(\neg \psi)$. Since $\Box_i B \subseteq s$, $\Box_i(\neg \psi) \in s$. $\neg \psi$ is equivalent to $\psi \rightarrow \perp$ and this means $\Box_i(\psi \rightarrow \perp) \in s$. By DISTV and $\forall \perp$, $\nabla_i(\psi, d) \in s$, contradicting to supposition that $\neg \nabla_i(\psi, d) \in s$.

Now we prove (!) by two cases:

Case 1: $A^+ \cup \overline{G(i, d)}$ is consistent. Then $A^+ \cup \overline{G(i, d)}$ can be extended by Lindenbaum Lemma to a maximal consistent set, say Γ . Let $\Gamma_1 = \Gamma_2 = \Gamma$, $t_1 = F(s, \Gamma_1, i, 0)$ and $t_2 = F(s, \Gamma_2, i, 1)$ and we have the following:

- $\psi \in t_1$, $\psi \in t_2$, $s \rightarrow_i t_1$, $s \rightarrow_i t_2$. By the construction method of F , this is immediate.
- $f_{t_1}(d) = 0$, $f_{t_2}(d) = 1$. From the construction rule of f in proposition 2, we can see that these are true, by the fact that for all $\phi \in \mathbf{LKv}^r$, either $\phi \notin \Gamma$ or $\nabla_i(\phi, d) \notin s$ and . In fact if $\nabla_i(\phi, d) \in s$, then $\phi \in G(i, d)$, $\neg \phi \in \overline{G(i, d)}$. This means $\neg \phi \in \Gamma$ and by the consistency of Γ , $\phi \notin \Gamma$

With the above facts, the (!) is obviously true now.

Case 2: $A^+ \cup \overline{G(i, d)}$ is inconsistent. Then there is a finite subset $\overline{G(i, d)}_0$ of $\overline{G(i, d)}$ and a finite subset A_0 of A such that $\vdash \bigwedge A_0 \wedge \psi \rightarrow \neg \bigwedge \overline{G(i, d)}_0$. Let $G(i, d)_0 = \{\chi \mid \neg \chi \in \overline{G(i, d)}_0\}$. By the fact that $\vdash \neg \bigwedge \overline{G(i, d)}_0 \leftrightarrow \bigvee G(i, d)_0$, we have $\vdash \bigwedge A_0 \wedge \psi \rightarrow \bigvee G(i, d)_0$. For convenience, name this formula δ_0

At this point, we need to split case 2 into two subcases, with the following proposition as the dividing line:

(*) for any $\chi_0 \in G(i, d)$ there is a $\chi \in G(i, d)$ such that $\chi \notin [\chi_0]_{i,d}$ and $A^+ \cup \{\chi\}$ is consistent.

Case 2.1: (*) is true. Since this still under Case 2, $A^+ \cup \overline{G(i, d)}$ is inconsistent, which implies that there is a $\chi_1 \in G(i, d)$ such that $A^+ \cup \{\chi_1\}$ is consistent (A^+ 's consistency is needed here). This implies, with (*), that there is a $\chi_2 \in G(i, d)$ such that $\chi_2 \notin [\chi_1]_{i,d}$ and $A^+ \cup \{\chi_2\}$ is consistent. The former means $\chi_1 \not\sim_{i,d} \chi_2$, thus $\nabla_i(\chi_1 \vee \chi_2, d) \notin s$, which in turn means $g_s(i, \chi_1, d) \neq g_s(i, \chi_2, d)$ by the definition 1. Now since $A \cup \{\chi_1\}$ and $A \cup \{\chi_2\}$ are both consistent, let Γ_1 and Γ_2 be the MCSs extended by them respectively, and $t_1 = F(s, \Gamma_1, i, d)$ and $t_2 = F(s, \Gamma_2, i, d)$. It is not hard to see that $f_{t_1}(d) = g_s(i, \chi_1, d) \neq g_s(i, \chi_2, d) = f_{t_2}(d)$, which justifies (!).

Case 2.2: (*) is false. Then the following

(**) there exists a $\chi_0 \in G(i, d)$ such that for any $\chi \in G(i, d)$, if $\chi \notin [\chi_0]_{i,d}$ then $A^+ \cup \{\chi\}$ is inconsistent.

is true. Under this supposition, let χ_0 be the element in $G(i, d)$ such that for any $\chi \in G(i, d)$. If $\chi \notin [\chi_0]_{i,d}$ then $A^+ \cup \{\chi\}$ is inconsistent. Further, let $[\chi_0]_{i,d}^0 = G(i, d)_0 \cap [\chi_0]_{i,d}$. Then, for any $\chi \in G(i, d)_0 \setminus [\chi_0]_{i,d}^0$, $\chi \notin [\chi_0]_{i,d}$, so $A^+ \cup \{\chi\}$ is inconsistent, which means $\vdash \bigwedge A'_0 \wedge \psi \rightarrow \neg \chi$

(note it as δ_χ) for some finite subset A'_0 of set A . Combining $\vdash \delta_\chi$ for all $\chi \in G(i, d)_0 \setminus [\chi_0]_{i,d}^0$, we have $\vdash \bigwedge A'_0 \wedge \psi \rightarrow \neg \bigvee (G(i, d)_0 \setminus [\chi_0]_{i,d}^0)$ again for some finite subset A'_0 of A . Note this long formula by δ_1 . Notice the following proof schema:

$$\begin{aligned} (1) \quad & \vdash \bigvee X \rightarrow (\bigvee Y \vee \bigvee (X \setminus Y)) \\ (2) \quad & \vdash \bigvee X \rightarrow (\neg \bigvee (X \setminus Y) \rightarrow \bigvee Y) \\ (3) \quad & \vdash (\bigvee X \wedge \neg \bigvee (X \setminus Y)) \rightarrow \bigvee Y \end{aligned}$$

Using this schema, and the fact that $\vdash \delta_0, \vdash \delta_1$, we have the following proof:

$$\begin{aligned} (4) \quad & \vdash \bigwedge A_0 \wedge \psi \rightarrow \bigvee G(i, d)_0 && [\text{this is } \delta_0] \\ (5) \quad & \vdash \bigwedge A'_0 \wedge \psi \rightarrow \neg \bigvee (G(i, d)_0 \setminus [\chi_0]_{i,d}^0) && [\text{this is } \delta_1] \\ (6) \quad & \vdash \bigwedge A_0 \wedge \bigwedge A'_0 \wedge \psi \rightarrow \bigvee G(i, d)_0 \wedge \neg \bigvee (G(i, d)_0 \setminus [\chi_0]_{i,d}^0) \\ (7) \quad & \vdash \bigwedge A_0 \wedge \bigwedge A'_0 \wedge \psi \rightarrow \bigvee [\chi_0]_{i,d}^0 \\ (8) \quad & \vdash \bigwedge A_0 \wedge \bigwedge A'_0 \rightarrow (\psi \rightarrow \bigvee [\chi_0]_{i,d}^0) \\ (9) \quad & \vdash \Box_i (\bigwedge A_0 \wedge \bigwedge A'_0) \rightarrow \Box_i (\psi \rightarrow \bigvee [\chi_0]_{i,d}^0) \end{aligned}$$

By definition of A and maximality, $\Box_i (\bigwedge A_0 \wedge \bigwedge A'_0) \in s$, so (***) : $\Box_i (\psi \rightarrow \bigvee [\chi_0]_{i,d}^0) \in s$.

Now we use a simple induction to show that $\nabla_i (\bigvee [\chi_0]_{i,d}^0, d) \in s$. Enumerate the formula in $[\chi_0]_{i,d}^0$ as $\lambda_1, \lambda_2, \dots, \lambda_n$ and inductively define $\Lambda_1 = \{\lambda_1\}, \Lambda_k = \Lambda_{k-1} \cup \{\lambda_k\}$.

Induction Hypothesis $g_s(i, \bigvee \Lambda_k, d) = g_s(i, \chi_0, d)$ and $\nabla_i (\bigvee \Lambda_k, d) \in s$.

Induction Basis $\lambda_1 \sim_{i,d} \chi_0$ so $\nabla_i (\lambda_1 \vee \chi_0, d) \in s$, then $g_s(i, \lambda_1, d) = g_s(i, \chi_0, d)$. Since $\lambda_1 \in G(i, d)$, $\nabla_i (\lambda_1, d) \in s$ automatically.

Induction Step For $\Lambda_k = \Lambda_{k-1} \cup \{\lambda_k\}$, firstly, by the same kind of argument in induction basis, $g_s(i, \lambda_k, d) = g_s(i, \chi_0, d)$. By IH, $g_s(i, \chi_0, d) = g_s(i, \bigvee \Lambda_{k-1}, d)$. So by the requirement (2) of a suitable possible canonical world in definition 1 imposed on s , $\nabla_i (\bigvee \Lambda_{k-1} \vee \lambda_k, d) = \nabla_i (\bigvee \Lambda_k, d) \in s$. Since $\vdash \lambda_k \rightarrow \bigvee \Lambda_k, \vdash \Diamond_i \lambda_k \rightarrow \Diamond_i \Lambda_k$. Yet $\lambda_k \in G(i, d)$ so $\Diamond_i \lambda_k \in s$, then $\Diamond_i \bigvee \Lambda_k \in s$. Then both $g_s(i, \lambda_k, d)$ and $g_s(i, \Lambda_k, d)$ are not *. So by (2) of definition 1 again, $g_s(i, \Lambda_k, d) = g_s(i, \lambda_k, d) = g_s(i, \chi_0, d)$.

By induction proof, $\nabla_i (\bigvee [\chi_0]_{i,d}^0, d) = \nabla_i (\bigvee \Lambda_n, d) \in s$. Then with DISTV and (***) we have proven, $\nabla_i (\psi, d) \in s$. But the proposition we intend to prove supposes $\neg \nabla_i (\psi, d) \in s$. Thus this case 2.2 is actually empty. \square

Now we are prepared to prove the truth lemma for \mathcal{M}^c :

Lemma 5 (truth lemma). *For any $s \in S^c$ and any $\phi \in LKv^r$, $\phi \in s$ iff $\mathcal{M}^c, s \Vdash \phi$.*

Proof. The inductive proof of this is a common practice in modal logic. Here we only show the two non-trivial cases:

$\phi = \Box_i \psi$ If $\Box_i \psi \in s$, then for any $t \in S^c$ such that $s \rightarrow_i t$, by the clause (3) of definition 1, $\psi \in t$, which by IH means $\mathcal{M}^c, t \Vdash \psi$. So $\mathcal{M}^c, s \Vdash \Box_i \psi$. For the other direction, suppose $\Box_i \psi \notin s$, then $\neg \Box_i \psi \in s$. By lemma 3 and IH, $\mathcal{M}^c, s \not\Vdash \Box_i \psi$.

$\phi = \nabla_i(\psi, d)$ If $\nabla_i(\psi, d) \in s$, then for any $t_1, t_2 \in S^c$ such that $s \rightarrow_i t_1, t_2$ and $\psi \in t_1, t_2$, by the clause (4) of definition 1, $f_{t_1}(d) = g_s(i, \psi, d) = f_{t_2}(d)$. For the other direction, suppose $\nabla_i(\psi, d) \notin s$, then $\neg \nabla_i(\psi, d) \in s$. By lemma 4 and IH, we have $t_1, t_2 \in S^c$ such that $s \rightarrow_i t_1, t_2$, $\mathcal{M}^c, t_1 \Vdash \psi$, $\mathcal{M}^c, t_2 \Vdash \psi$ and $f_{t_1}(d) \neq f_{t_2}(d)$. So $\mathcal{M}^c, s \not\models \nabla_i(\psi, d)$.

□

Based on this, we are able to present:

Theorem 6. \mathbf{LKv}^r is sound and strongly complete for \mathbf{LKv}^r .

Proof. Soundness is rather simple. For any consistent set $\Delta \subseteq \mathbf{LKv}^r$, using Lindenbaum Lemma for \mathbf{LKv}^r , there exists a **MCS** Γ such that $\Delta \subseteq \Gamma$. Now let f be a constant function from \mathbf{D} to 0, and g be defined in the exactly same fashion as in proposition 2. According to definition 1, $s = \langle \Gamma, f, g \rangle \in S^c$, so by truth lemma, for any $\phi \in \Delta$, $\mathcal{M}^c, s \Vdash \phi$ and thus Δ is satisfiable. Then strong completeness follows. □

4 Complexity

In this section, we will give a **PSPACE** algorithm in light of tableau method for the satisfiability problem of \mathbf{LKv}^r . Since \mathbf{LKv}^r contains **K**, the lower bound is also **PSPACE**. So we can conclude that the decision problem of \mathbf{LKv}^r is **PSPACE**-complete.

4.1 Rules of tableau

Definition 7. A propositional tableau is a set of formula X satisfying the following:

- if $\neg\neg\phi \in T$ then $\phi \in T$,
- if $\neg(\phi \wedge \psi) \in T$ then $\neg\phi \in T$ or $\neg\psi \in T$,
- if $\phi \wedge \psi \in T$ then $\phi \in T$ and $\psi \in T$,
- if $\phi \in T$ then $\neg\phi \notin T$ and vice versa,

We call a violation of the last clause “blatantly inconsistent”. X is fully expanded if and only if for any $\phi \in X$ and ψ a subformula of ϕ , either ψ or $\neg\psi$ is in X .

Definition 8. A state is a tuple $\langle X, g, h, ha, hb \rangle$ satisfying:

- X is a fully expanded propositional tableau.
- Let $E_X = \{\langle i, d \rangle \mid \text{for some } \phi, \nabla_i(\phi, d) \in X\}$, $G_X(i, d) = \{\phi \mid \nabla_i(\phi, d) \in X\}$, $E_X(i) = \{d \mid \langle i, d \rangle \in E_X\}$.
- g is a function defined on set E_X . $g(i, d)$ is a 2-tuple $\langle A, B \rangle$ such that:

- $A \subseteq G_X(i, d), B \subseteq \mathcal{P}(G_X(i, d));$
- $A \cup \bigcup B = G_X(i, d), A \cap \bigcup B = \emptyset;$
- B is a partition of $\bigcup B$, always including empty set;

In the sequel let $g(i, d)[1]$ denote such A and $g(i, d)[2]$ for such B .

- h is a function defined on set $\{\langle i, \phi \rangle \mid \neg \Box_i \phi \in X\}$. $h(i, \phi)$ is again a function defined on $E_X(i)$. For every $d \in E_X(i)$, $h(i, \phi)(d) \in g(i, d)[2]$.
- ha, hb are both function defined on set $\{\langle i, \phi, d \rangle \mid \neg \nabla_i(\phi, d)\}$. $ha(i, \phi, d)$ and $hb(i, \phi, d)$ are again functions defined on set $E_X(i) \cup \{d\}$ such that for d' in their domain:
 - if $d' \in E_X(i)$, then $g(i, d')$ is defined, and $ha(i, \phi, d)(d') \in g(i, d')[2], hb(i, \phi, d)(d') \in g(i, d')[2];$
 - if $d' \notin E_X(i)$, then $d' = d$. In this case $ha(i, \phi, d)(d') = hb(i, \phi, d)(d') = \emptyset;$
 - either $ha(i, \phi, d)(d) \neq hb(i, \phi, d)(d)$ or both of them are \emptyset .

As we did in the proof of completeness, these functions g, h, ha, hb are also “extra information”. The function g here is actually a enumeration of all possible equivalence relation $\sim_{i,d}$ given in the proof of proposition 2.

It is worthwhile here to briefly discuss the number of possible g, h, ha, hb for a given X . Obviously $|E_X|, |E_X(i)|, |G_X(i, d)| \leq |X|$. For function g , note that A and B together forms a partition of $|G_X(i, d)|$. So the cardinality of the range of g is at most $|X|^{|X|}$. Since the domain of g is E_X , the cardinality of the domain of g is at most $|X|$. Thus the total number of possible g is at most $|X|^{|X|^{|X|}} = |X|^{|X|^2}$. Similarly, the number of all possible h, ha and hb are bounded by $|X|^{|X|}$. Summing all these together, given X , the number of all possible $\langle g, h, ha, hb \rangle$ is at most $|X|^{|X|^2+3 \times |X|}$.

Now we present the method of deciding the satisfiability of a **LKv**^r formula ϕ_0 trough building a tree. In the following rules, L means the formula set of a node, F represents the additional information needed (g, h, ha, hb) , and C is a partial function from $D(\phi_0)$ to \mathbb{Z} represents the required assignments of value names occurred in ϕ_0 . Since the set of all finite subsets of a countable set is also countable, there is a function, say, $code(X)$ to code each finite set of formulas into a unique positive integer.

1. Construct a tree with a single node s_0 as its root, and let $L(s_0) = \{\phi_0\}, F(s_0) = \emptyset, C(s_0) = \emptyset$.
2. Repeatedly try each of following rules in their order until none of them applies:
 - (a) **Forming propositional tableau:** if s is a leaf node, $L(s)$ is not blatantly inconsistent and not a propositional tableau, then there must be a $\psi \in L(s)$ such that following 3 rules applies:
 - i. if $\psi = \neg \neg \chi$, add a new node s' and an edge between s and s' to the tree (i.e. a successor of s), and set $L(s') = L(s) \cup \{\chi\}, F(s') = F(s), C(s') = C(s)$.
 - ii. if $\psi = \neg(\chi_1 \wedge \chi_2)$, add two successor s_1, s_2 of s , and set $L(s_i) = L(s) \cup \{\neg \chi_i\}, F(s_i) = F(s), C(s_i) = C(s)$ for $i = 1, 2$.

- iii. if $\psi = \chi_1 \wedge \chi_2$, add a successor s' of s and set $L(s') = L(s) \cup \{\chi_1, \chi_2\}$, $F(s') = F(s)$, $C(s') = C(s)$.
- (b) **Forming fully expanded propositional tableau:** if s is a leaf node, $L(s)$ is a propositional tableau but not a fully expanded propositional tableau, then there must be $\phi \in Sub(L(s))$ such that ϕ and $\neg\phi$ are both not in $L(s)$. In this case add two successor s_1, s_2 of s and set $L(s_1) = L(s) \cup \{\phi\}$, $L(s_2) = L(s) \cup \{\neg\phi\}$, $F(s_1) = F(s_2) = F(s)$, $C(s_1) = C(s_2) = C(s)$.
- (c) **forming state:** if s is a leaf node, $L(s)$ is a fully expanded propositional tableau, but $\langle L(s), F(s) \rangle$ is not a state, then for all function tuple F' such that $\langle L(s), F' \rangle$ is a state, add a successor s' to s and set $L(s') = L(s)$, $F(s') = F'$, $C(s') = C(s)$. Notice that the total number of such F' is bounded by $|\phi_0|^{|L(s)|^2 + 3 \times |L(s)|}$, as argued above.
- (d) **Add labeled successors:** if s is a leaf node, $\langle L(s), F(s) \rangle$ is a state and in $L(s)$ there are at least one formula of the form $\neg\Box_i\phi$ or $\neg\nabla_i(\phi, d_0)$, then there should be some labeled successors to s :

- For each ϕ such that $\neg\Box_i\phi \in L(s)$, add an i -successor(i.e. with an edge labeled i) s' to s and set $L(s') =$

$$\{\neg\phi\} \cup L(s) \setminus \Box_i \cup \bigcup_{d \in E_{L(s)}(i)} \neg(g_s(i, d)[1] \cup \bigcup g_s(i, d)[2] \setminus h_s(i, \phi)(d))$$

$C(s') = h_s(i, \phi)$ and $F(s')$ all empty functions.

- For each ϕ such that $\neg\nabla_i(\phi, d_0) \in L(s)$, add two i -successor s_a and s_b to s and for $\mathbf{x} = a, b$, set $L(s_{\mathbf{x}}) =$

$$\{\phi\} \cup L(s) \setminus \Box_i \cup \bigcup_{d \in E_{L(s)}(i)} \neg(g_s(i, d)[1] \cup \bigcup g_s(i, d)[2] \setminus h_{\mathbf{x}_s}(i, \phi, d_0)(d))$$

Set $C(s_{\mathbf{x}}) = h_{\mathbf{x}_s}(i, \phi, d_0)$ for $\mathbf{x} = a, b$. If $h_{s_a}(i, \phi, d_0)(d_0) = h_{s_b}(i, \phi, d_0)(d_0)$ then change $C(s_a)(d_0)$ to \bullet and $C(s_b)(d_0)$ to \circ . Finally set $F(s_{\mathbf{x}}) = \emptyset$.

- (e) **Mark satisfiable:** if s is not yet marked, non of the above three rules applies, and all its successors(possibly none) have been marked, then:
 - if the edges to the successors of s are not labeled, then mark s as "satisfiable" if any one of its successors is marked "satisfiable", otherwise mark "unsatisfiable".
 - if the edges to the successors of s are labeled, then mark s as "satisfiable" if all of its successors are marked "satisfiable", otherwise mark "unsatisfiable".
 - if s has no successors, then mark s as "satisfiable" if $L(s)$ is not blatantly inconsistent, otherwise mark "unsatisfiable".

3. if root s_0 is marked "satisfiable" then return ϕ_0 is satisfiable, otherwise ϕ_0 is unsatisfiable.

Lemma 9. For any LKv^r formula ϕ_0 , the tree construction method defined above terminates.

Proof. It is immediate to see that if s' is a successor of s generated by rule (1) or (2) then $L(s) \subsetneq L(s')$ but for all s in the tree, $L(s) \subseteq Sub^+(\phi_0)$. If s' is generated from s by rule (3), then rule (1) (2) and (3) are no longer applicable to s' . This means the longest chain of unlabeled edges will not exceed $2 \times |\phi_0| + 1$ otherwise there must be a blatant inconsistency. At the same time, if s' is generated from s by rule (4), then $depth(L(s')) < depth(L(s))$. Thus in any branch

the number of labeled edges will not exceed $|\phi_0|$. So we can conclude that the depth of the tree is bounded by $2 * |\phi_0|^2$. On the other hand, the branching number for any node is also bounded by $|\phi_0|^{|\phi_0|^2 + 3 * |\phi_0|}$. So this construction must terminate. \square

After proving that this tableau must halt, the correctness of this tableau must be argued for now. Correctness means that, root s_0 is marked “satisfiable” if and only if ϕ_0 is satisfiable. The following two lemmas present two directions of correctness respectively.

Lemma 10. *For any \mathbf{LKv}^r formula ϕ_0 , if after the tree construction defined above, root s_0 is marked “satisfiable”, then ϕ_0 is satisfiable.*

Proof. Suppose the root is marked “satisfiable”. Then we can build a model satisfying ϕ_0 from the constructed tree. Let $\mathcal{M} = \langle W, O, \{\rightarrow_i \mid i = 1..n\}, V, V_D \rangle$ where:

- $W = \{s \mid s \text{ is marked “satisfiable” and } \langle L(s), F(s) \rangle \text{ is a state}\};$
- $O = \text{all finite subset of } \mathbf{LKv}^r \text{ plus } \bullet \text{ and } \circ;$
- $s \rightarrow_i t$ if and only if there exists $s' \in W$ such that s' is an i -successor of s and t is reachable from s through a sequence of unlabeled edges;
- for all $s \in W$, if $p \in L(s)$ then $s \in V(p)$, if $\neg p \in L(s)$ then $s \notin V(p)$;
- for all $s \in W$, if $C(s)(d)$ is defined, then $V_D(d, s) = C(s)(d)$.

By our construction method, there must be such a model. Now we can prove that if $\phi \in L(s)$ then $\mathcal{M}, s \models \phi$ by a induction on $Sub^+(\phi_0)$. We give the key step of that induction:

- if $\nabla_i(\phi, d) \in L(s)$, then $d \in E_{L(s)}(i)$ and $\phi \in G_{L(s)}(i, d)$. Since $\langle L(s), F(s) \rangle$ is a state, $g_s(i, d)$ satisfies the clauses in the definition of state. Particularly, $\phi \in G_{L(s)}(i, d) = g_s(i, d)[1] \cup \bigcup g_s(i, d)[2]$. Consider following two cases:
 - if $\phi \in g_s(i, d)[1]$, then by restraints on g_s and ha_s and rule (d), it is immediate that for all i -successors of s s' , $\neg\phi \in L(s')$. Thus for all s'' reachable from s' through a sequence of unlabeled edges, $\neg\phi \in L(s'')$. So if $s \rightarrow_i t$, $\neg\phi \in L(t)$. By induction hypothesis, $\mathcal{M}, t \not\models \phi$. Thus $\nabla_i(\phi, d)$ is trivially true on s .
 - if $\phi \in \bigcup g_s(i, d)[2]$, then there is a unique $X \in g_s(i, d)[2]$ such that $\phi \in X$. Now for any t such that $s \rightarrow_i t$, by the property of \rightarrow_i , there exists s' such that s' is an i -successor of s and t is reachable from s' through a sequence of unlabeled edges. By rule (d), s' must be generated by a formula of the form $\neg\Box_i\psi$ or $\neg\nabla_i(\psi, d_0)$. W.l.o.g we suppose it is generated by $\neg\nabla_i(\psi, d_0)$ and ha_s . If $\phi \in L(s')$, then $\neg\phi \notin L(s')$, because s' must be marked “satisfiable” and thus is not blatantly inconsistent. Again by rule (d), $\phi \in ha_s(i, \psi, d_0)(d)$ for if not so, ϕ will in $\bigcup g_s(i, d)[2] \setminus ha_s(i, \psi, d_0)(d)$, then $\neg\phi$ will be in $L(s')$, contradiction. By the constraints on ha_s , $ha_s(i, \psi, d_0)(d)$ must be X . Then by rule (d) again, $C(s')(d) = X$ and thus $C(t)(d) = X$. With this frame of argument, we can conclude that for all t such that $s \rightarrow_i t$, if $\phi \in L(t)$ then $C(t)(d) = X$. By induction hypothesis ($\mathcal{M}, t \models \phi$ implies $\phi \in L(t)$) and restraint on V_D , we can conclude that $\mathcal{M}, s \models \nabla_i(\phi, d)$.

- if $\neg\nabla_i(\phi, d) \in L(s)$, then it is immediate from rule (d) that there are two i -successor s_a and s_b such that $\phi \in L(s_a), \phi \in L(s_b), C(s_a) \neq C(s_b)$. Since s is marked “satisfiable”, s_a and s_b must also be so. By rule (e) and the finiteness of this tree, there must be t_a, t_b in W and reachable through a sequence of unlabeled edges from s_a and s_b respectively. Then $\phi \in L(t_a)$ and $\phi \in L(t_b)$ and $C(t_a)(d) = C(s_a)(d) \neq C(s_b)(d) = C(t_b)(d)$. By induction hypothesis and \mathcal{M} ’s properties, $s \rightarrow_i t_a, s \rightarrow_i t_b, \mathcal{M}, t_a \models \phi, \mathcal{M}, t_b \models \phi, V_D(d, t_a) \neq V_D(d, t_b)$. So $\mathcal{M}, s \models \neg\nabla_i(\phi, d)$.
- If $\mathcal{M}, s \models \nabla_i(\phi, d)$ then $\nabla_i(\phi, d) \in L(s)$. For suppose not, then $\neg\nabla_i(\phi, d) \in L(s)$, then $\mathcal{M}, s \models \neg\nabla_i(\phi, d)$, contradiction. Similar results goes for $\neg\nabla_i(\phi, d)$.

Since the root is marked “satisfiable”, there must be a s reachable through unlabeled edges from s_0 such that $s \in W$. Then $\phi_0 \in L(s)$ and then $\mathcal{M}, s \models \phi_0$, so ϕ_0 is satisfiable. \square

Lemma 11. *If ϕ_0 is satisfiable, then after the construction for ϕ_0 , root s_0 will be marked “satisfiable”.*

Proof. Through a induction from leaves to roots, we show that if $\langle L(s), F(s) \rangle$ is not a state and $L(s)$ is satisfiable, then s is marked “satisfiable”.

First, if s is a leaf, and $L(s)$ is satisfiable, then $L(s)$ must not be blatantly inconsistent. But since s is a leaf, this suffices for s to be marked “satisfiable”.

If s is not a leaf and rule (a) or (b) was applied to s : w.l.o.g we show the case where (b) was applied to s , generating successor s_1 and s_2 . Suppose both $L(s_1)$ and $L(s_2)$ are unsatisfiable, then by completeness theorem we have shown, they are inconsistent. So $\phi_{L(s)} \rightarrow \phi$ and $\phi_{L(s)} \rightarrow \neg\phi$ are derivable. Thus $\phi_{L(s)} \rightarrow \perp$ is derivable, $L(s)$ is inconsistent. By soundness, $L(s)$ is unsatisfiable. Take a contraposition, we have if $L(s)$ is satisfiable, then either $L(s_1)$ or $L(s_2)$ is satisfiable. By induction hypothesis (note that s_1 and s_2 are not states), either s_1 or s_2 is marked “satisfiable”. By rule (e), s is marked “satisfiable”.

If s is not a leaf and rule (c) was applied to s : suppose $L(s)$ is satisfiable, let $\mathcal{M}, s = \langle W, O, \{\rightarrow_i \mid i = 1, \dots, n\}, V, V_D \rangle, s$ be the model that satisfies $L(s)$. Now let g be a function on $E_{L(s)}$ such that:

- $g(i, d)[1] = \{\phi \in G_{L(s)}(i, d) \mid \text{for all } t \text{ such that } s \rightarrow_i t, \mathcal{M}, t \not\models \phi\}$;
- $g(i, d)[2]$ is the partition of set $\{\phi \in G_{L(s)}(i, d) \mid \text{there exist } t : s \rightarrow_i t, \mathcal{M}, t \models \phi\}$ defined by relation \sim where $\psi_1 \sim \psi_2$ if and only if there exists t_1, t_2 such that $s \rightarrow_i t_1, s \rightarrow_i t_2, \mathcal{M}, t_1 \models \psi, \mathcal{M}, t_2 \models \psi, V_D(d, t_1) = V_D(d, t_2)$. This \sim relation is evidently a equivalence relation. Let $f(i, d, x)$ be the unique set $X \in g(i, d)[2]$ such that there exists $\psi \in X$ and $t \in W$ such that $\mathcal{M}, t \models \psi$ and $V_D(d, t) = x$. If there is no such a X in $g(i, d)[2]$, let $f(i, d, x) = \emptyset$.

Then, let h be a function on $\{\langle i, \psi \rangle \mid \neg\Box_i\psi \in L(s)\}$. By supposition, $\mathcal{M}, s \models \neg\Box_i\psi$ for any i, ψ in the domain of h . This means there exists $t \in W$ such that $s \rightarrow_i t$ and $\mathcal{M}, t \not\models \psi$. Now let $h^*(i, \psi) = t$ and $h(i, \psi)$ be a function on $E_{L(s)}(i)$ such that $h(i, \psi)(d) = f(i, d, V_D(d, t))$. Further, let h_a, h_b be functions on $\{\langle i, \psi, d_0 \rangle \mid \neg\nabla_i(\psi, d_0) \in L(s)\}$. By supposition, $\mathcal{M}, s \models \neg\nabla_i(\psi, d_0)$ for any $\langle i, \psi, d_0 \rangle$ in the domain of h_a and h_b . This means there exists $t_a, t_b \in W$ such that both of them is accessible from s through i , satisfies ψ but $V_D(d_0, t_a) \neq V_D(d_0, t_b)$. Now for

$x = a, b$, let $hx^*(i, \psi, d_0) = t_x$ and let $hx(i, \psi, d_0)$ be a function on $E_{L(s)}(i) \cup \{d_0\}$ such that $h(i, \psi, d_0)(d) = f(i, d, V_D(d, t_x))$ for $d \in E_{L(s)}(i)$ and if $d_0 \notin E_{L(s)}(i)$, set $hx(i, \psi, d_0)(d_0) = \emptyset$.

Now it is evident that $\langle L(s), g, h, ha, hb \rangle$ is a state, so by rule (c), there will be a successor s' of s such that $F(s') = \langle g, h, ha, hb \rangle$. It is also not hard to see that after applying rule (d) to s' , for every successor s'' of s' , $L(s'')$ is satisfiable, because if s'' is generated by $h(i, \psi)$ (or $ha(i, \psi, d_0)$, $hb(i, \psi, d_0)$), then $\mathcal{M}, h^*(i, \psi)$ (or $ha^*(i, \psi, d_0)$, $hb^*(i, \psi, d_0)$) $\models L(s'')$.

To see this more clearly, suppose s'' is generated by $ha(i, \psi, d_0)$ and let $t = ha^*(i, \psi, d_0)$. If $\alpha \in L(s'')$, then by rule (d), there are several cases:

- $\alpha = \psi$. By selection of $ha^*(i, \psi, d_0)$, this is evident;
- $\alpha \in L(s) \setminus \Box_i$. Since $s \rightarrow_i t$, this is also evident;
- for some $d \in E_{L(s)}(i)$, $\alpha \in \neg g(i, d)[1]$. By definition of $g(i, d)[1]$, every i -accessible world from s refutes every formula in $g(i, d)[1]$. So $\mathcal{M}, t \models \alpha$.
- for some $d \in E_{L(s)}(i)$, $\alpha \in \neg(\bigcup g(i, d)[2] \setminus ha(i, \psi, d_0)(d))$. Let $\alpha = \neg\beta$. Then β does not belong to $ha(i, \psi, d_0)(d)$. Towards a contradiction suppose that $\mathcal{M}, t \models \beta$, then by definition of f , $f(i, d, V_D(d, t)) = [\beta]_{\sim}$. By definition of ha , $ha(i, \psi, d_0)(d) = [\beta]_{\sim}$, so $\beta \in ha(i, \psi, d_0)(d)$, contradiction. Thus, $\mathcal{M}, t \models \neg\beta$.

In conclusion, there is a successor s' of s such that for every successor s'' of s' , $L(s'')$ is satisfiable. By induction hypothesis, all such s'' is marked “satisfiable”. By rule (e), s' is marked “satisfiable”, and so is s . \square

It is straightforward to turn above construction method into an algorithm running in polynomial space, using a depth-first search. For stepping down in the search tree, we need to record where we are currently by a stack where in every level a set of subformulas of ϕ_0 is kept and the height of this stack is at most $|\phi_0|^2$. Thus we need $\mathcal{O}(|\phi_0|^2 \times |\phi_0|)$ space. As the width of this tableau is exceedingly large, extra space is needed for branching. We need to enumerate all possible F properly. At each level of the stack, we need to record where we are when enumerating F so that the next F can be calculated. This consumes $\mathcal{O}(|\phi_0|^2 \times |\phi_0|^2)$ space. This means this algorithm runs in $\mathcal{O}(|\phi_0|^4)$ space, that is, in **PSPACE**. Since this logic also contains modal logic **K**, its satisfiability problem is **PSPACE**-hard. So we have theorem:

Theorem 12. *The satisfiability problem for logic LKv^r is **PSPACE**-complete.*

5 Conclusion

In this paper, we showed that LKv^r is sound and complete for LKv^r over arbitrary models and gave a tableau for this logic. This starts the study of the complexity of similar “knowing what” logic.

Our proof of completeness is relatively simpler than its counterpart in [11]. Exactly what makes this possible needs further investigation, and we conjecture that if this cause can be found, we may give a beautiful frame of completeness proof upon which proving completeness on a special model class will become much easier.

Our tableau is not simple, and more importantly, unlike tableaux for normal modal logic where if a formula is unsatisfiable, a proof of its negation can be effectively constructed, our tableau for \mathbf{LKv}^r cannot provide this proof now. This commands further study, but our conjecture here is that a proof of the negation of a unsatisfiable formula is attainable from this tableau or its slightly tweaked version, even though it is not found yet.

The complexity of \mathbf{ELKv}^r is what attracted us initially, and our tableau may shed some light on it. Yet it is still arguable that whether it is in \mathbf{PSPACE} . To make things more explicit, we should try adding formulas $d = x$ and $\Box_i(\phi \rightarrow d = x)$ directly into tableau instead of using $G_i(\phi, d)$ and partitions which may only work on model class \mathbf{K} .

Last but not least, we should consider extending our language to incorporate more first-order characteristics, such as predicate or equality. If such extension does not bring too much complexity or other undesirable property, we may also try to give a good logic on encryption, as Cohen and Dam did in [1].

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